A model of damage mechanics for the deformation of the continental crust

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1986, Turcotte

[1] We derive a continuum rheology for the deformation of the continental crust using continuum damage mechanics. It is hypothesized that when a constant strain rate $\dot{\varepsilon}$ is applied to a solid material, the stress $\sigma$ and damage increase until failure occurs, which is analogous to an earthquake. We further assume that this process is repeated, in analogy to the reoccurrence of earthquakes on a fault. Our model assumes that the continental crust behaves elastically below a yield stress $\sigma_y$. Above this stress the continuum deformations can be modeled as a non-Newtonian viscous flow with $\dot{\varepsilon} \propto (\sigma - \sigma_y)^n$, where $n$ is constant. We derive the modified Omori’s law for aftershock decay using a viscoelastic version of our model and get good agreement with observations taking $n = 6$. Using parameter values appropriate for aftershocks, we obtain a continuum crustal rheology that can explain major orogenies such as the Indian-Asian collision.


1. Introduction

[2] Brittle and ductile deformation plays major roles in the deformation of the continental crust. Block models and continuum models are associated with brittle and ductile processes, respectively, and have been proposed as a means of quantifying the deformation of the continental crust. Significant continental deformation takes place on major faults. Models in which deformation occurs on a specified array of faults are the block models. King et al. [1994a], Thatcher [1995], and Jackson [2002] have discussed the merits of a block (microplate) description of active continental tectonics. However, the frequency-size distribution of faults is fractal so that we can conclude that faults are present on a wide range of scales. Tectonic models for fractal distributions of faults have been proposed by King [1983, 1986], Turcotte [1986], King et al. [1988], and Hirata [1989].

[1] An alternative approach to the deformation of the continental crust is to use continuum models. In these models ductile deformation is assumed and creep (Newtonian and non-Newtonian) and plastic rheologies are utilized. To describe the ductile behavior of rocks at high temperature, an empirical power law equation between strain rate $\dot{\varepsilon}$ and stress $\sigma$ has been proposed as

$$\dot{\varepsilon} = A\sigma^n \exp \left( \frac{-Q + PV_e}{RT} \right),$$

where $A$ and $n$ are the constants, $Q$ is the activation energy, $V_e$ is the activation volume, $R$ is the gas constant, $T$ is the absolute temperature and $P$ is the pressure [e.g., Ashby and Verall, 1977; Kirby and Kronenberg, 1987; Karato and Wu, 1993]. This empirical relation is called Dorn’s equation for the steady state flow of a solid [e.g., Dorn 1954; Nicolas and Poirier, 1976] and it is valid for both diffusion creep ($n = 1$) and dislocation creep ($n = 3–5$). An example of a continuum model for the deformation of the continental crust is the viscous sheet model where equations similar to equation (1) were used [e.g., Bird and Piper, 1980; England and McKenzie, 1982; Houseman and England, 1986, 1993; England and Houseman, 1986]. However, these studies did not consider the role of brittle processes in the deformation of the continental crust.

[4] Both ductile and brittle processes can also be quantified using the concept of dislocations. Dislocation theories have been developed to explain not only the mechanical properties of crystals but also their optical and electromagnetic properties. Taylor [1934], Orowan [1934], and Polanyi [1934] applied the concept of dislocations to the plastic deformation of simple crystals in order to explain why the observed yield stresses of crystals are much lower than the

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theoretical values calculated from atomic theory assuming a perfect lattice state. Mura [1969] developed a method of continuously distributed dislocations to consider the relationship between macroscopic plasticity and dislocation theory. However, these studies have not considered how microcracks contribute to the deformation of solids.

We hypothesize that deformation in the continental crust is dominated by displacements on faults at all scales. Our objective is to derive a continuum rheology that is applicable to this deformation. An avenue for irreversible behavior associated with these brittle and ductile processes is damage mechanics. The concept of damage mechanics has been utilized widely in engineering problems [Krajcinovic, 1996; Skrzypek and Ganczarski, 1999; Voyiadis and Kattan, 1999].

Damage refers to irreversible deformation of solids. Some examples include plasticity, brittle microcracking, thermally activated creep and so on [Krajcinovic, 1996]. In order to quantify the brittle deformation of solids associated with microcracking, an empirical continuum model of damage mechanics was introduced and is widely used in civil and mechanical engineering [Kachanov, 1986]. This type of model is often called continuum damage model and has also been applied to tectonic processes [Lyakhovsky et al., 1997; Ben-Zion and Lyakhovsky, 2002; Scherbakov et al., 2005]. Another approach to the anelastic deformation of composite materials is provided by the fiber-bundle model, a discrete model of damage mechanics. The fiber-bundle model was applied to fatigue in structural materials and earthquakes in geophysical settings [e.g., Hemmer and Hansen, 1992; Moreno et al., 2001]. The continuum damage model was shown to be equivalent to the fiber-bundle model for the occurrence of a failure in a simple geometry [Krajcinovic, 1996; Turcotte et al., 2003]. The concept of the yield stress was used for damage mechanics by Scherbakov and Turcotte [2003, 2004]. These authors considered that damage occurs in the form of microcracking at stresses greater than the yield stress.

In this paper we show that when the continuum damage model is applied to the brittle deformation of the continental crust, a non-Newtonian power law viscous rheology is obtained. In our analyses a yield stress is introduced as follows: below this stress the continental crust behaves elastically and can act as a stress guide, above the yield stress the continuum deformations can be modeled as a power law viscous fluid. This expands upon the previous work of Turcotte and Glasscoe [2004] that did not consider a yield stress.

Another consequence of the viscoelastic behavior of the crust is the occurrence of aftershock sequences. A main shock suddenly increases the stress in some regions of the crust adjacent to the fault on which the main shock occurred. Stress relaxation is accompanied by the aftershock sequences. The modified Omori’s law [Utsu, 1961] gives a temporal decay in the rate of aftershock occurrence [e.g., Scholz, 2002]. We show that the rate of energy release in our model corresponds to the rate of aftershock decay described by the modified Omori’s law.

2. Continuum Damage Model

Utilizing a continuum damage model, we will show that the result obtained is identical to a non-Newtonian fluid rheology. We assume a linear dependence of the stress in a solid material on time at a constant strain rate $\dot{\varepsilon}$. The stress and damage increase until failure occurs, which is analogous to an earthquake. The process is then repeated: this allows us to model the recurrences of earthquakes on a fault. To model damage evolution from undamaged to damaged state, we introduce a yield stress $\sigma_y$ and corresponding yield strain $\varepsilon_y$. If the stress is less than the yield stress $\sigma < \sigma_y$, there is no damage and linear elasticity is applicable. If the stress is greater than the yield stress $\sigma > \sigma_y$, damage occurs and a state variable $\alpha$ can be introduced to model this irreversible behavior.

We first consider that a brittle solid obeys linear elasticity for stresses in the range $0 \leq \sigma \leq \sigma_y$. We also assume that Hooke’s law is applicable so that the dependence of stress $\sigma$ on strain $\varepsilon$ is given by

$$\sigma = E_0\varepsilon, \quad (2)$$

where $E_0$ is Young’s modulus, a constant. From equation (2) the corresponding yield strain $\varepsilon_y$ is given by

$$\varepsilon_y = \frac{\sigma_y}{E_0}. \quad (3)$$

We further assume that damage occurs in the form of microcracks at stresses greater than the yield stress $\sigma > \sigma_y$. As a consequence the behavior of the solid will deviate from that predicted by linear elasticity. Following Scherbakov et al. [2005], we introduce a damage variable $\alpha$ according to

$$\varepsilon - \varepsilon_y = \frac{\sigma - \sigma_y}{E_0(1 - \alpha)}. \quad (4)$$

The damage variable $\alpha$ quantifies deviations from linear elasticity and the distribution of microcracks in the material considered. By definition the damage variable $\alpha$ takes values between 0 and 1 ($0 \leq \alpha \leq 1$) and failure occurs when $\alpha = 1$. Damage evolution is a transient process so that we have $\alpha(t)$ and the damage variable increases as long as $\sigma > \sigma_y$ or until failure occurs ($\alpha = 1$). The following two examples are representative for damage evolution. If a constant stress $\sigma_0 > \sigma_y$ is applied instantaneously, microcracking will occur and $\alpha$ increases until failure occurs at $\alpha = 1$. If constant strain $\varepsilon_0 > \varepsilon_y$ is applied the stress will relax until $\sigma = \sigma_y$, with again $\alpha = 1$.

On the basis of thermodynamic considerations [Kachanov, 1986; Krajcinovic, 1996; Lyakhovsky et al., 1997], the time evolution of the damage variable $\alpha$ was related to stress $\sigma$ and strain $\varepsilon$. Scherbakov and Turcotte [2003, 2004] and Scherbakov et al. [2005] modified their results to include a yield stress $\sigma_y$ and corresponding yield strain $\varepsilon_y$. In this formulation the time evolution of damage is given by

$$\frac{d\alpha}{dt} = A(\sigma)\left(\frac{\varepsilon}{\varepsilon_y} - 1\right)^2, \quad (5)$$

where

$$A(\sigma) = 0 \quad \text{if} \quad 0 \leq \sigma \leq \sigma_y, \quad (6)$$

$$A(\sigma) = \frac{1}{\eta} \left(\frac{\sigma}{\sigma_y} - 1\right)^p \quad \text{if} \quad \sigma > \sigma_y, \quad (7)$$
with \( t_d \) the characteristic timescale for damage and \( \rho \) a power law exponent to be determined from experiments. \[ [11] \] For strains \( \varepsilon \) greater than \( \varepsilon_y \) and corresponding stresses \( \sigma \) greater than \( \sigma_y \), we substitute equations (4) and (7) into equation (5) and obtain

\[
\frac{d\alpha}{dt} = \frac{1}{t_d} \left( \frac{\sigma}{\sigma_y} - 1 \right)^{\rho+2} \frac{1}{(1 - \alpha)^2}. \tag{8}
\]

We assume that the applied stress increases linearly with time. With the initial condition of stress \( \sigma = \sigma_y \) at \( t = 0 \), we assume that the stress at subsequent times is given by

\[
\sigma = \sigma_y + E_0 \dot{\varepsilon} t. \tag{9}
\]

Substituting equation (9) into equation (8) using equation (3), we obtain

\[
\frac{d\alpha}{dt} = \frac{1}{t_d} \left( \frac{\dot{\varepsilon}}{\varepsilon_y} \right)^{\rho+2} \frac{1}{(1 - \alpha)^2}. \tag{10}
\]

Integrating with the initial condition \( \alpha = 0 \) when \( t = 0 \), we find

\[
\alpha = 1 - \left\{ 1 - \frac{3}{2} \left( \frac{\dot{\varepsilon}}{\varepsilon_y} \right)^{\rho+2} \left( \frac{\sigma}{\sigma_y} \right)^{\rho+3} \right\}^{\frac{1}{\rho+3}}. \tag{11}
\]

Failure occurs at time \( t_f \) when \( \alpha = 1 \), thus we have

\[
t_f = \left\{ \frac{(\rho + 3)t_d}{3} \right\}^{\frac{2}{\rho+3}} \left( \frac{\sigma_y}{\varepsilon_y} \right)^{\frac{\rho+3}{\rho+2}}. \tag{12}
\]

The failure time \( t_f \) increases with the inverse of the strain rate \( \dot{\varepsilon} \) for large values of the power law exponent \( \rho \). From equation (9) the mean stress during the run-up to failure is given by

\[
\sigma = \sigma_y + \frac{1}{2} E_0 \dot{\varepsilon} t_f. \tag{13}
\]

Furthermore, substitution of equations (3) and (12) into equation (13) gives

\[
\frac{\dot{\varepsilon}}{\varepsilon_y} = \frac{3}{t_d(\rho + 3)} \left\{ 2 \left( \frac{\sigma}{\sigma_y} - 1 \right) \right\}^{\frac{\rho+3}{\rho+2}}. \tag{14}
\]

It is convenient to introduce a new exponent \( n = \rho + 3 \) and a new characteristic time

\[
\tau_c = \frac{nt_d}{3\varepsilon_y \left( \frac{\sigma_y}{2E_0} \right)^n}. \tag{15}
\]

In terms of this characteristic time \( \tau_c \), we can rewrite equation (14) as

\[
\dot{\varepsilon} = \frac{1}{\tau_c} \left( \frac{\sigma - \sigma_y}{E_0} \right)^n. \tag{16}
\]

So far we have considered the failure of a single element. In order to model the continuum deformation of the continental crust we assume that a failed element is immediately replaced by a new undamaged element with the stress in the element equal to the yield stress. The repetitive failure of the element is hypothesized to model the reoccurrence of earthquakes on a fault. The failure stress is equivalent to the static frictional stress on faults and the yield stress is equivalent to the stress which is assumed to be the stress on the fault after an earthquake. We assume that the continental crust is made up of many independent units, and each of these units experiences the periodic failures considered above. The failures on a unit are the periodic earthquakes that occur on a fault or fault segment.

\[ [14] \] The application of equation (16) to the continental crust results in a non-Newtonian viscous rheology in terms of the relationship between strain rate \( \dot{\varepsilon} \) and the excess stress over the yield stress \( \sigma - \sigma_y \). Many authors have applied a non-Newtonian viscosity to the deformation of the continental crust [e.g., Bird and Piper, 1980; England and McKenzie, 1982; Housman and England, 1986, 1993; England and Housman, 1986]. However, many plate interiors do not appear to exhibit any internal deformation. Thus the inclusion of a yield stress in any crustal rheology appears to be a requirement. No continuum deformation occurs if the stress is less than the yield stress.

\[ [15] \] Turcotte and Glasscoe [2004] modeled the repetitive failure of faults using a fiber-bundle model. They also derived a non-Newtonian viscous rheology but their analysis did not include a yield stress. The similarity of results using fiber-bundle and continuum damage models is not surprising since they can be shown to be equivalent under some conditions [Turcotte et al., 2003].

\[ [16] \] It is of interest to consider the microphysics of the rheological model presented above. We relate this physics to time delays associated with brittle failure as the time evolution of damage plays an important role in our rheological model. We explore these time delays from the viewpoint of metastable phase changes. In the derivation of a nonlinear viscous rheology for the continental crust given above, the characteristic timescale for damage \( t_d \) plays a crucial role. What is the physical basis for introducing this timescale? Although the damage mechanics model has been shown empirically to quantify the time delays associated with brittle failure, no rigorous derivation of the model to explain the time delays has been provided. The time delays associated with the nucleation and growth of a single crack are often attributed to stress corrosion [Freund, 1990]. This explanation has also applied to the time delays associated with aftershocks [Das and Scholz, 1981].

\[ [17] \] More recently, a number of statistical physicists have related time delays associated with brittle fracture to the time delays associated with metastable phase changes. Buchel and Sethna [1997], Zapperi et al. [1997], and Kun and Herrmann [1999] have drawn an analogy between brittle fracture and a first-order phase transition. Sornette and Andersen [1998] and Gluzman and Sornette [2001] argue that brittle rupture is analogous to a critical point (a second-order phase change). Selinger et al. [1991], Rundle et al. [1999, 2000], and Zapperi et al. [1999] have drawn an analogy between brittle failure and spinodal nucleation. The basic idea is that there is an analogy between the nucleation
of bubbles in a superheated liquid and the nucleation of microcracks in a brittle solid.

[18] The analogy between phase transitions and fracture also has a thermodynamic basis. Thermal fluctuations are crucial in phase transitions of solids and liquids. A fundamental question is whether temperature plays a significant role in the damage of brittle materials. Guarino et al. [1989] varied the temperature in their experiments on the fracture of chipboard and found no effect. However, a systematic temperature dependence has been found for the lifetime statistics of Kevlar fibers [Wu et al., 1988].

[19] Time delays associated with bubble nucleation in a superheated liquid are explained in terms of thermal fluctuations. The fluctuations must become large enough to overcome the stability associated with surface tension in a bubble. The fundamental question in damage mechanics is the cause of the delay in the occurrence of damage. This problem has been considered in some detail by Ciliberto et al. [2001] and Scorretti et al. [2001]. These authors attributed damage to the “thermal” activation of microcracks. An effective “temperature” can be defined in terms of the spatial disorder (heterogeneity) of the solid. The spatial variability of stress in the solid is caused by the microcracking itself, not by thermal fluctuations. This microcracking occurs on a wide range of scales.

[20] A related time delay is associated with friction. Laboratory studies of friction between rock surfaces are empirically correlated with rate-and-state friction laws [Dieterich, 1978; Ruina, 1983]. When the slip velocity is changed there is a time delay associated with the establishment of a new steady state coefficient of friction. The state variable in the friction law has also been used to quantify the delay in rupture on a frictional surface. Dieterich [1994] has derived Omori’s law for the time delay of earthquake aftershocks from the equations of rate-and-state friction. There is clearly a close association between this work and the damage mechanics approach.

3. Viscoelasticity

[21] In the previous section we derived a continuum fluid rheology for the continental crust which is non-Newtonian power law with a yield stress. It is useful for a number of problems to combine a fluid rheology on long timescales with an elastic behavior on short timescales. For this purpose a viscoelastic rheology is usually used [Turcotte and Schubert, 2002]. The Maxwell model for viscoelasticity considers a material in which the total strain rate \( \dot{\epsilon} \) is hypothesized to be the sum of an elastic strain rate \( \dot{\epsilon}_{el} \) and a viscous strain rate \( \dot{\epsilon}_v \),

\[
\dot{\varepsilon} = \dot{\epsilon}_{el} + \dot{\epsilon}_v = \frac{\dot{\sigma}}{E_0} + \frac{\dot{\sigma}}{\mu}, \tag{17}
\]

where \( \mu \) is the viscosity of the material. If a constant strain \( \varepsilon_0 \) is applied instantaneously the resulting stress will relax in a viscoelastic relaxation time \( \tau = \mu E_0 \). For the Earth’s mantle we take \( E_0 = 7 \times 10^{10} \) Pa and \( \mu = 10^{21} \) Pa s so that \( \tau = 450 \) years. The simple Maxwell model combines the short timescale elastic behavior associated with seismic waves with geological timescale fluid behavior associated with mantle creep and convection.

\[ \dot{\varepsilon}_{el} = \frac{\dot{\sigma}}{E_0}, \]

(18)

If the stress on the medium is less than the yield stress \( \sigma < \sigma_y \) there is no viscous deformation and the material behaves elastically. If the stress on the medium is greater than the yield stress \( \sigma > \sigma_y \) viscous strain will occur and the viscous strain rate is given by equation (16) as

\[
\dot{\varepsilon}_v = \frac{1}{\mu} \left( \frac{\sigma - \sigma_y}{E_0} \right)^n. \tag{19}
\]

The total strain rate \( \dot{\varepsilon} \) in equation (17) is the sum of the strain rate \( \dot{\varepsilon}_v \) from equation (19) and the time derivative \( \dot{\varepsilon}_{el} \) of equation (18), thus we have

\[
\dot{\varepsilon} = \frac{1}{\mu} \frac{d\sigma}{dt} + \frac{1}{\tau} \left( \frac{\sigma - \sigma_y}{E_0} \right)^n. \tag{20}
\]

This is the rheological law relating strain rate, stress, and rate of change of stress for our Maxwell viscoelastic material. Following Turcotte and Schubert [2002], we assume that the deformation of the continental crust can be treated as the behavior of the Maxwell viscoelastic material.

4. Constant Applied Strain

[21] As a first example, we consider the response of the viscoelastic medium to a sudden application of constant strain \( \varepsilon_0 \) at time \( t = 0 \), the strain is maintained constant for \( t > 0 \). As discussed above, the behavior of the material is elastic during the very rapid application of strain. Therefore the initial stress \( \sigma_0 \) is

\[
\sigma_0 = E_0 \varepsilon_0. \tag{21}
\]

If \( \varepsilon_0 < \varepsilon_y \) no damage occurs and the initial stress remains unchanged \( \sigma = \sigma_0 \) for \( t > 0 \). If \( \varepsilon_0 > \varepsilon_y \), the material is strained elastically along the path ABD in Figure 1. The strain is then maintained constant \( \varepsilon_0 \) so that damage evolves and repetitive material failures occur. Because of the damage and failures, the stress on the sample relaxes from the initial stress \( \sigma_0 \) to the yield stress \( \sigma_y \). This relaxation takes place along the path DFG illustrated in Figure 1. The solution will give the time dependence of stress \( \sigma(t) \) during the stress relaxation.

[24] For \( t > 0 \) \( \dot{\varepsilon} = 0 \) and equation (20) reduces to

\[
0 = \frac{1}{E_0} \frac{d\sigma}{dt} + \frac{1}{\tau} \left( \frac{\sigma - \sigma_y}{E_0} \right)^n. \tag{22}
\]

Integrating with the initial condition \( \sigma = \sigma_0 \) at \( t = 0 \), we find

\[
\frac{\sigma - \sigma_y}{\sigma_0 - \sigma_y} = \left\{ 1 + (n - 1) \left( \frac{\sigma_0 - \sigma_y}{E_0} \right)^{n-1} \frac{t}{\tau} \right\}^{1/n}. \tag{23}
\]
In the limit $t \to \infty$, $\pi \to \sigma_y$ and no further damage can occur, as expected. Before discussing the application of this solution to earthquake aftershocks, we discuss the temporal decay of aftershock activity.

The modified Omori’s law [Utsu, 1961] describes the temporal decay of aftershock activity and is given in the form [Scholz, 2002]

$$\frac{dN}{dt} = \frac{K}{(c + t)^p}.$$  \hspace{1cm} (24)

where $dN/dt$ is the rate of occurrence of aftershocks with magnitudes greater than $m$, $t$ is the time that has elapsed since the main shock and $K$, $p$ and $c$ are parameters. This law is a manifestation of temporal correlations in aftershock sequences which can be viewed as complex relaxation processes occurring after main shocks. Omori [1894] introduced equation (24) with $p = 1$, usually a good approximation. For $t \gg c$ the dependence of $dN/dt$ on $t$ in equation (24) gives a power law behavior for aftershock decay. This type of temporal fractal property is called a long time tail [Takayasu, 1990].

Shcherbakov et al. [2005] derived an alternative form for the temporal decay of aftershock activity. They called this a generalized Omori’s law. They obtained it from a combination of the Gutenberg-Richter relation [Gutenberg and Richter, 1954], Bâth’s law [Bâth, 1965], and the modified Omori’s law given in equation (24) with the result

$$\frac{1}{N_f} \frac{dN}{dt} = \frac{p - 1}{c} \frac{1}{(1 + t/c)^p},$$  \hspace{1cm} (25)

where $N_f$ is the total number of aftershocks with magnitudes greater than $m$. In order for this relation to be valid it is necessary that $p > 1$.

We now model the time-dependent decay of aftershocks using our viscoelastic analysis. Following Shcherbakov and Turcotte [2003, 2004] and Shcherbakov et al. [2005], our working hypothesis is that stress transfer during a main shock increases the stress and strain above the yield values $\sigma_y$ and $e_y$ in some regions adjacent to the fault on which the main shock occurred. The increases of stress and strain are essentially instantaneous and follow linear elasticity. We believe that it is a good approximation to neglect any increase in regional stress due to tectonics during the aftershock sequence. We hypothesize that the applied strain $\varepsilon_0$ remains constant and that aftershocks relax the stress to its yield value $\sigma_y$. The occurrence of aftershocks is attributed to this relaxation process as given in equation (23).

In order to quantify the rate of aftershock occurrence we determine the rate of energy release in the relaxation process. The stored elastic energy density (per unit mass) $e_0$ in a material after an instantaneous strain $\varepsilon_0$ has been applied along the path ABD in Figure 1 is

$$e_0 = \frac{1}{2} E_0 \varepsilon_0^2.$$  \hspace{1cm} (26)

Since the strain is constant during the stress relaxation, no work is done on the sample. If the applied strain (stress) is instantaneously removed at point $F$ we hypothesize that the sample will follow the elastic path FH which is parallel to...
the path ABD. The elastic energy $e_1$ recovered during the stress relaxation on this path is given by

$$e_1 = \frac{1}{2} \pi \varepsilon_0. \quad (27)$$

Substitution of equation (23) into equation (27) with equations (3) and (21) gives

$$e_1 = \frac{1}{2} E_0 \varepsilon_0 (\varepsilon_0 - \varepsilon_y) + \frac{1}{2} E_0 \varepsilon_0 (\varepsilon_0 - \varepsilon_y) \frac{1}{1 + (n-1)(\varepsilon_0 - \varepsilon_y)^{n-1} \left( \frac{t}{\tau_c} \right)} \quad (28).$$

We assume that the difference between the energy added $e_0$ and the energy recovered $e_1$ is lost in aftershocks. This energy $e_{AF}$ corresponds to the area ABDFH in Figure 1 and is given by subtracting the energy recovered $e_1$ in equation (28) from the energy added $e_0$ in equation (27) as

$$e_{AF} = e_0 - e_1 = \frac{1}{2} E_0 \varepsilon_0 (\varepsilon_0 - \varepsilon_y) \left[ 1 - \frac{1}{1 + (n-1)(\varepsilon_0 - \varepsilon_y)^{n-1} \left( \frac{t}{\tau_c} \right)} \right] \quad (29).$$

The total energy of aftershocks $e_{AFT}$ is obtained in the limit $t \to \infty$ with the result

$$e_{AFT} = \frac{1}{2} E_0 \varepsilon_0 (\varepsilon_0 - \varepsilon_y). \quad (30)$$

Taking the time derivative of equation (29) and using equation (30) we obtain the rate of energy release

$$\frac{1}{e_{AFT}} \frac{de_{AFT}}{dt} = \frac{1}{t c} (\varepsilon_0 - \varepsilon_y)^{n-1} \left( \frac{t}{\tau_c} \right) \quad (31)$$

This relation is clearly very similar to the generalized Omori’s law given in equation (25).

Since Gutenberg-Richter frequency statistics is applicable to aftershocks it is appropriate to assume

$$\frac{1}{N_t} \frac{dN}{dt} = \frac{1}{e_{AFT}} \frac{de_{AFT}}{dt}. \quad (32)$$

With this condition equations (25) and (31) are identical if we take

$$n = \frac{p}{p - 1}, \quad (33)$$

$$c = \frac{\tau_c}{(n-1)(\varepsilon_0 - \varepsilon_y)^{n-1}} \left( \frac{t}{\tau_c} \right). \quad (34)$$

Substituting equations (33) and (34) into equation (31) with equation (32) gives

$$\frac{1}{N_t} \frac{dN}{dt} = \frac{1}{e_{AFT}} \frac{de_{AFT}}{dt} = \frac{p - 1}{c} \frac{1}{1 + t/c}. \quad (35)$$

We will now consider the applicable values of $c$ and $p$ for aftershock sequences and derive the applicable values of $n$, $\tau_c$, and $\varepsilon_0 - \varepsilon_y$ with

$$\Delta \sigma \equiv \sigma_0 - \sigma_y = E_0 (\varepsilon_0 - \varepsilon_y). \quad (36)$$
Shcherbakov et al. [2005] have studied aftershock sequences following four California earthquakes: Landers (magnitude $M_M = 7.3$), Northridge ($M_M = 6.7$), Hector Mine ($M_M = 7.1$) and San Simeon ($M_M = 6.5$). These authors determined the rates of occurrence of aftershocks with magnitudes greater than $m$ in number per day as a function of time for 1460 days (100 days for San Simeon earthquake) after a main shock. They fit equation (25) to the data and the $c$ values were obtained for lower cutoff magnitudes $m = 1.5, 2.0, 2.5, 3.0, 3.5, 4.0$. They showed that the $c$ values are not constant but scale with the lower cutoff magnitude $m$. Their values for $c$ for the four aftershock sequences are given in Figure 2. These authors found $p = 1.22$ for Landers, $p = 1.18$ for Northridge, $p = 1.21$ for Hector Mine, and $p = 1.08$ for San Simeon.

For our further analyses we take the typical value $p = 1.2$ and from equation (33) we find $n = 6$. Following Turcotte and Schubert [2002], we take $E_0$ to be $5 \times 10^{10}$ Pa for crustal deformation. We further assume that $\tau_c = 10^{-20}$ s and will discuss this choice in some detail in section 5. The corresponding values of $\Delta \sigma = \sigma_0 - \sigma_y$ and $\varepsilon_0 - \varepsilon_y$ are given in Figure 3 as a function of the characteristic time $c$ from equations (34) and (36). Also included in Figure 3 are the values of $c$ given in Figure 2 for the four earthquakes.

We create Figure 4 based on the values of $m$ given in Figure 2 and the values of $\Delta \sigma = \sigma_0 - \sigma_y$ and $\varepsilon_0 - \varepsilon_y$ given in Figure 3. Dependence of the excess stress $\Delta \sigma = \sigma_0 - \sigma_y$ and excess strain $\varepsilon_0 - \varepsilon_y$ on the characteristic decay time $c$ from equations (34) and (36) taking $\tau_c = 10^{-20}$ s, $E_0 = 5 \times 10^{10}$ Pa, and $p = 1.2 (n = 6)$. Also shown are the characteristic decay times for the four earthquakes as given in Figure 2.

Figure 3. Dependence of the excess stress $\Delta \sigma = \sigma_0 - \sigma_y$ and excess strain $\varepsilon_0 - \varepsilon_y$ on the characteristic decay time $c$ from equations (34) and (36) taking $\tau_c = 10^{-20}$ s, $E_0 = 5 \times 10^{10}$ Pa, and $p = 1.2 (n = 6)$. Also shown are the characteristic decay times for the four earthquakes as given in Figure 2.

Figure 4. The results from Figures 2 and 3 are combined so that required excess stress $\Delta \sigma = \sigma_0 - \sigma_y$ and excess strain $\varepsilon_0 - \varepsilon_y$ are given as a function of the lower-magnitude aftershock cutoff $m$. 

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in Figure 3. The values of the excess stresses $\Delta \sigma = \sigma_0 - \sigma_y$ and strains $\varepsilon_0 - \varepsilon_y$ are given as a function of the lower-magnitude cutoff for aftershocks $m$. The values of the excess stress appear to be reasonable. We see that the excess stress increases with the magnitude of the aftershocks. This can be attributed to the association between higher stress levels and larger aftershocks. It is reasonable that the large aftershocks occur in regions where the stress transfer from the main shock is large. We also see for the four cases considered that the stresses required to generate aftershocks are less for larger earthquake than for smaller earthquakes. One explanation for these results is that the high amplitude surface waves generated by large earthquakes weaken the faults in which aftershocks occur.

5. Constant Applied Stress

We next consider the behavior of our viscoelastic medium if a constant stress $\sigma_0 > \sigma_y$ is applied at $t = 0$ and maintained constant for $t > 0$. During the very rapid application of stress the material behaves elastically along the path ABD in Figure 1. Therefore we have the initial strain $\varepsilon_0$ at $t = 0$ as given by equation (21). Subsequently, there is no change in the stress, $d\sigma/dt = 0$ along the path DE illustrated in Figure 1. Equation (20) reduces to

$$\dot{\varepsilon}_v = \frac{1}{\tau_c} \left( \frac{\sigma_0 - \sigma_y}{E_0} \right) \varepsilon_v^n.$$  \hfill (37)

We get a power law relationship between the strain rate $\dot{\varepsilon}_v$ and the excess of stress $\sigma_0 - \sigma_y$. Our Maxwell model represents a non-Newtonian viscous fluid rheology with a stress threshold $\sigma_y$. Equation (37) can be integrated with equations (3) and (21) and the initial condition $\varepsilon = \varepsilon_0$ at $t = 0$ to give

$$\varepsilon_v = \frac{1}{\tau_c} \left( \frac{\sigma_0 - \sigma_y}{E_0} \right)^n \varepsilon_0.$$  \hfill (38)

We obtain a linear dependence of the total strain $\varepsilon_v$ on time $t$.

We next apply the derived result in equation (37) to the continuum deformation of the continental crust. We take $E_0 = 5 \times 10^{10}$ Pa and $n = 6$ as determined in the previous section. The rheology is determined when the characteristic time $\tau_c$ is specified.

For the continuum deformation of the continental crust we expect the stress $\sigma - \sigma_y$ to be in the range of $10^{-3}$ to $10^1$ MPa and the strain rate $\dot{\varepsilon}_v$ to be in the range of $10^{-18}$ to $10^{-10}$ s$^{-1}$. We take the characteristic time $\tau_c = 10^{-28}$, $10^{-24}$, $10^{-20}$, $10^{-16}$, and $10^{-12}$ s. The resulting dependence of stress $\sigma - \sigma_y$ on strain rate $\dot{\varepsilon}_v$ is given in Figure 5. It is seen that the stress varies little with increasing strain rate. If there were no dependence, constant stress independent of strain rate, the rheology would be perfectly plastic.

Also included for comparison is the dependence of stress on strain rate for a linear Newtonian viscous rheology,

$$\sigma - \sigma_y = \eta \dot{\varepsilon}_v,$$  \hfill (39)

where $\eta$ is the viscosity. The range of equivalent crustal viscosities is from $\eta = 10^{10}$ to $\eta = 10^{25}$ Pa s. There is a strong dependence of $\sigma - \sigma_y$ on $\dot{\varepsilon}_v$. With the nonlinear viscous rheology with large power law exponent the behavior of the deforming upper continental crust approaches that of a perfect plastic material. A similar discussion was given by Turcotte and Glasscoe [2004], but this did not include the stress threshold.

Although the continuum deformation of the continental crust can be associated with a wide range of stresses and strain rates, a representative value would be $\sigma - \sigma_y = 0.1$ MPa and $\dot{\varepsilon}_v = 10^{-14}$ s$^{-1}$; these values correspond to the solid circle in Figure 5. The characteristic time that gives these values is $\tau_c = 10^{-20}$ s. This value was the value used for the deformation of the continental crust in the previous section. Thus the same characteristic time $\tau_c = 10^{-20}$ s gives
reasonable values for the aftershock relaxation, i.e., weeks to months and also reasonable values for the long-term crustal deformation, i.e., millions of years.

6. Discussion

[39] At low confining pressure rock behaves as a brittle material; that is, it fractures when a large stress is applied. However, when the confining pressure approaches a rock's brittle strength, the yield stress can be defined in a transition from brittle or elastic behavior to ductile or plastic behavior. We introduced the concept of yield stress into our continuum-damage model. Our model with yield stress is applicable to the deformation of the "damaged" continental crust that is loaded at high-confining pressures. This stress was not included into previous analyses [Turcotte and Glasscoe, 2004].

[40] The large value of the power law exponent (n = 6) is certainly not surprising for orogenic zones. Houseman and England [1986] considered an indenter model for continental deformation and applied the finite element method to thin non-Newtonian viscous sheet to obtain solutions. Using the power law rheology given in equation similar to equation (37), they obtained results for n = 2 and 9. England and Houseman [1986] compared these results with observations in the Indian-Asian collision zone and found broad agreement provided that the power law exponent is large (n > 2). Our continuum rheology can explain the brittle deformation in orogenies such as this collision.

[41] The values of the excess stress are consistent with other results. To understand whether Landers earthquake changed the proximity to failure on San Andreas Fault system, King et al. [1994b] explored how changes in Coulomb conditions associated with one or more earthquakes may trigger subsequent events. These authors found the distribution of aftershocks for Landers earthquake, as well as for several other events in its vicinity, can be explained by the Coulomb stress as follows: aftershocks are abundant where the Coulomb stress on optimally oriented faults rose by more than 0.15 MPa, and aftershocks are sparse where the Coulomb stress dropped by a similar amount. If the stress in our model is comparable to the Coulomb stress on optimally oriented faults, this result of raised stress is in agreement with our excess stress given in Figures 3 and 4.

[42] Shcherbakov et al. [2005] studied the stress relaxation process in a continuum damage model and found that the rate of energy release in the model is identical to the rate of aftershocks described by the generalized Omori formula in equation (25). We used the time evolution of damage in equations (5), (6), and (7) that is the same as that given by Shcherbakov et al. [2005] and found that our result is identical to their result. One aspect of damage mechanics that was considered in our paper but not by Shcherbakov et al. [2005] is "healing." If a material heals, then the damage and the damage variable decrease. When studying material failure it is not necessary to consider healing. However, any steady state deformation of a brittle material requires both the generation and healing of the damage. The continental crust is by definition a damaged, brittle material. Earthquakes associated with displacements on faults are analogous to the acoustic emissions from microcracking during the failure of brittle solid. However, earthquakes are repetitive so that quasi steady state deformations of the continental crust can occur. This requires a balance between damage, the creation of new faults and increased displacements on existing faults, and the "healing" of faults. In this paper we considered the repetitive occurrence of earthquakes. This was modeled by the increase of the damage variable, from zero (undamage) to one (failure), and the repetition of the process. This repetition includes material healing. We derived an applicable rheology in equation (16) for the continuum deformation of the continental crust. Therefore, in order for a quasi steady state behavior, our model requires that active faults must become inactive, i.e., they must "heal".

[43] Some forms of "damage" that we did not considered in this paper are clearly thermally activated. The deformation of solids by diffusion and dislocation creep is an example. The ability of vacancies and dislocations to move through a crystal is governed by an exponential dependence on absolute temperature. Another example is given by Nakatani [2001], who documented a systematic temperature dependence of rate and state friction. Sornette and Ouillon [2005] used thermally activated rupture process to find that seismic decay rates after main shocks follow the modified Omori law in equation (24). The continuum deformation of the continental crust has already been considered as a thermally activated process by Nanjo and Turcotte [2005] utilized the fiber-bundle model. These authors assumed that fiber failure is a thermally activated earthquake. They obtained a power law relation between stress and strain rate and the rheology is exponentially dependent on the inverse absolute temperature as given by equation (1). Their analyses based on laboratory experiments [e.g., Nakatani, 2001] argued in favor of thermally activated damage in order to find the strength envelope of the continental lithosphere.

[44] However, it is a matter of controversy whether temperature plays a significant role in the damage of brittle failure of materials. Guarino et al. [1998] varied the temperature in their experiments on the fracture of chipboard and found no effect.

[45] In this paper, we showed that the continuum damage model gives non-Newtonian viscous fluid behavior above the yield stress. Below the yield stress elastic deformation was assumed. Applying this model to viscoelastic problems, we considered crustal deformation. In our model the deformation of the continental crust above the yield stress is due to the repetitive occurrence of faulting on a wide range of scales. The model reproduces the temporal power law decay of aftershock occurrence associated with stress relaxation after a main shock. With parameter values appropriate for our aftershock observation, the model provides a continuum rheology applicable to the brittle deformation in orogenic zones.

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