Appendix: Property of $F(m)$ and its relations to model criticality

It is evident from (16) that $\beta > \alpha$ is required. As it is shown later, the influences of these 3 parameters, A, α and β, to ζ and F is mainly due to their influences to the critical parameter ϱ. Here we are going to discuss $F(m)$ under all the three cases: 1° subcritical case, where each family tree dies off finally and the whole process is stable and stationary; 2° critical case, where each family tree dies off with a long tail and the population of the whole process in unit time increases unboundedly; and, 3°, supercritical case, where some of the family trees may never die off and the population of whole process will explode.

Function $F(m)$ is also closely related to the extinct probability of the family tree starting from an event of magnitude m, namely $P_c(m)$, which can be derived in the following way.

$$\begin{aligned}
P_c(m) &= \mathbf{P}\{\text{The family tree from an event of magnitude } m \text{ extinguishes}\} \\
&= \mathbf{P}\{\text{an event of magnitude } m \text{ produces finite number of offspring}\} \\
&= \sum_{n=0}^{\infty} \mathbf{P}\{\text{each child produces finite offspring} \mid m \text{ has } n \text{ children}\} \\
&\qquad \times \mathbf{P}\{m \text{ has } n \text{ children}\} \\
&= \sum_{n=0}^{\infty} \left[\int_{m_c}^{+\infty} s(m^*) P_c(m^*)\,\mathrm{d}m^*\right]^n \frac{[\kappa(m)]^n}{n!} \mathrm{e}^{-\kappa(m)} \\
&= \exp\left[-\kappa(m)\left(1 - \int_{m_c}^{+\infty} s(m^*) P_c(m^*)\,\mathrm{d}m^*\right)\right]. \qquad (22)
\end{aligned}$$

Substitute $P_c(m) = \exp[-C\kappa(m)]$ into (22), we have

$$C = 1 - \int_{m_c}^{+\infty} s(m^*) \exp[-C\kappa(m^*)]\,\mathrm{d}m^*. \qquad (23)$$

Substitute (8) and (2) into (23),

$$C = 1 - \frac{\beta}{\alpha} C^{-\frac{\beta}{\alpha}} \Gamma\left(-\frac{\beta}{\alpha}, C\right). \qquad (24)$$

Compare (23) to (11),

$$\lim_{m \to +\infty} F(m) = C = -\frac{\log P_c(m)}{\kappa(m)}. \qquad (25)$$

It is easy to prove that (23) has one solution C in $(0,1)$ if and only if the processes is supercritical, i.e., $\varrho = \int_{m_c}^{\infty} \kappa(m)\, s(m)\, \mathrm{d}m > 1$.

For the subcritical case, which requires $\beta > \alpha$ and $\varrho = A\beta/(\beta-\alpha) < 1$, it is easy to see that $F(m) \to 0$ when $m \to +\infty$ because $C = 0$. That is to say, when the process is subcritical the larger the event, the less chance that it has a larger descendant. To discuss how fast F tends to 0, it is useful to use the following approximation. If φ is not an integer,

$$\begin{aligned} \Gamma_\varphi(x_1) - \Gamma_\varphi(x_2) &= \int_{x_1}^{x_2} u^{\varphi-1}\mathrm{e}^{-u}\,\mathrm{d}u \\ &= \sum_{n=0}^{+\infty} \frac{(-1)^n \left(x_2^{n+\varphi} - x_1^{n+\varphi}\right)}{n!(n+\varphi)}; \end{aligned} \tag{26}$$

if $\varphi = -k$ is a non-positive integer, we can replace the kth item in the summation by $(-1)^k \log(x_2/x_1)/k!$. Equations (14) and (26) give

$$F(m) = 1 - \frac{\beta}{\alpha}[A\,F(m)]^{\frac{\beta}{\alpha}} \sum_{n=0}^{+\infty} (-1)^n X_n(m), \tag{27}$$

where

$$X_n(m) = \begin{cases} \dfrac{[A\,F(m)]^{n-\frac{\beta}{\alpha}} \left(\mathrm{e}^{(n\alpha-\beta)(m-m_c)} - 1\right)}{n!\left(n - \frac{\beta}{\alpha}\right)}, & \text{for } n \neq \beta/\alpha, \\ \dfrac{\alpha(m-m_c)}{n!}, & \text{for } n = \beta/\alpha. \end{cases} \tag{28}$$

Now reconsider the behavior of the solution F under different conditions for the parameters. Formally expanding the exponential in the integral on the right side of (27) and setting $m_c = 0$ to abbreviate the notation, we obtain

$$F(m) = e^{-\beta m} + \frac{A\beta}{\beta-\alpha} F(m)[1 - e^{(\alpha-\beta)m}] + \frac{A^2\beta}{2(2\alpha-\beta)} F^2(m)\left[1 - e^{(2\alpha-\beta)m}\right] + \cdots. \tag{29}$$

Since $\frac{A\beta}{\beta-\alpha} = \rho$, this reduces to

$$(1-\rho)F(m) = e^{-\beta m} - \rho\, F(m)\, e^{(\alpha-\beta)m} + \frac{A^2\beta}{2(2\alpha-\beta)} F^2(m)\left[1 - e^{(2\alpha-\beta)m}\right] + \cdots. \tag{30}$$

When the process is subcritical, because we are looking for the solution of $F(m)$ such that $F(m)\,\mathrm{e}^{\alpha(m-m_c)} \to 0$ when $m \to \infty$, (30) can approximately by keeping the first two terms, which gives

$$\lim_{m\to+\infty} \frac{F(m)}{s(m)} = \frac{1}{\beta(1-\varrho)}. \tag{31}$$

If $\rho = 1$ (critical case) the left side of (30) vanishes. We have then $A = 1 - \alpha/\beta$, and we may write

$$0 = 1 + \mathrm{e}^{\alpha m} F(m) + \frac{A^2 \beta}{2(2\alpha - \beta)} F^2(m)\, \mathrm{e}^{\beta m} \left[1 - e^{(2\alpha-\beta)m}\right] + \cdots . \qquad (32)$$

Here some care is needed to sort out the leading terms. For $\beta/2 < \alpha < \beta$, it can be claimed that a solution exists with leading term

$$F(m) = \mathrm{e}^{-\alpha m}[1 + o(1)].$$

Under these conditions the order F term in (32) remains of order 1 while $F^2(m)\mathrm{e}^{\beta m} \to 0$.

However, in the case $\alpha \leq \beta/2$, it is the F^2 term which dominates. We claim that a solution exists in which the leading term has the form

$$F(m) = A^{-1}\sqrt{2}\,(1 - 2\alpha/\beta)^{-\frac{1}{2}}\, \mathrm{e}^{-\beta m/2}$$

for the F term in (32) is then order $\mathrm{e}^{-(\beta/2-\alpha)m}$, and converges to zero, as does $F^2\, \mathrm{e}^{\beta m}\, \mathrm{e}^{(2\alpha-\beta)m}$, while $F^2\, \mathrm{e}^{\beta m}$ remains bounded, leading to a solution of the form claimed.

In the first of the two cases, ζ is approximately a function of $m - m'$, whereas in the second case it is a function of $\alpha m - \beta m'/2$. The first form holds in situations where the family size grows rather quickly with the parent's magnitude m', while the second form holds only in situations where the growth is relatively slow. Both forms of ζ are illustrated in the diagrams.

When the process is supercritical, $\varrho > 1$ and $C > 0$. Equation (25) yields

$$\lim_{m\to\infty} F(m) = C, \qquad (33)$$

implying that $F(m)$ tends to a positive constant when m is sufficiently large. That is, the probability that the population of the family tree be infinite is greater than 0 when the process is supercritical.